

# On the arc-analytic type of some weighted homogeneous polynomials

Jean-Baptiste Campesato <sup>\*</sup>

December 24, 2016

## Abstract

It is known that the weights of a complex weighted homogeneous polynomial  $f$  with isolated singularity are analytic invariants of  $(\mathbb{C}^d, f^{-1}(0))$ . When  $d = 2, 3$  this result holds by assuming merely the topological type instead of the analytic one.

G. Fichou and T. Fukui recently proved the following real counterpart: the blow-Nash type of a real singular non-degenerate convenient weighted homogeneous polynomial in three variables determines its weights.

The aim of this paper is to generalize the above-cited result with no condition on the number of variables. We work with a characterization of the blow-Nash equivalence called the arc-analytic equivalence. It is an equivalence relation on Nash function germs with no continuous moduli which may be seen as a semialgebraic version of the blow-analytic equivalence of T.-C. Kuo.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Recollection</b>	<b>3</b>
2.1	The arc-analytic equivalence . . . . .	3
2.2	A motivic invariant of the arc-analytic equivalence . . . . .	3
<b>3</b>	<b>The modified zeta function of a non-degenerate polynomial</b>	<b>5</b>
<b>4</b>	<b>Application to convenient non-degenerate weighted homogeneous polynomials</b>	<b>7</b>
4.1	Computation of $B_f(T)$ . . . . .	7
4.2	Computation of $A_f(T)$ . . . . .	8
4.3	Proof of the invariance of the weights . . . . .	9

<b>References</b>	<b>15</b>
-------------------	-----------

## 1 Introduction

K. Saito [28] proved that the weights of a weighted homogeneous polynomial  $f$  with isolated singularity are analytic invariants of  $(\mathbb{C}^d, f^{-1}(0))$ . Then it was proved for  $d = 2$  and  $d = 3$ , respectively by E. Yoshinaga and M. Suzuki [30] and by O. Saeki [27] that the weights are also determined by the topological type of  $(\mathbb{C}^d, f^{-1}(0))$ .

T. Fukui [13, Conjecture 9.2] conjectures the real counterpart for the blow-analytic equivalence of T.-C. Kuo [19]. Do two quasihomogenous polynomials with isolated singularity which are blow-analytic equivalent share the same weights?

2010 Mathematics Subject Classification. 14P20 (14B05 32S15 14E18).

<sup>\*</sup> Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Sakura-ku, Saitama 338-8570, Japan.

E-mail address: [jbcampesato@mail.saitama-u.ac.jp](mailto:jbcampesato@mail.saitama-u.ac.jp)

Research supported by a Japan Society for the Promotion of Science (JSPS) Postdoctoral Fellowship (Short-term) for North American and European Researchers.

In the two variable case, the positive answer has been given by O. M. Abderrahmane [1] for weighted homogeneous polynomials non-degenerate with respect to their Newton polyhedron.

Then G. Fichou and T. Fukui [12] proved the conjecture for non-degenerate convenient weighted homogeneous polynomials in three variables up to the blow-Nash equivalence.

The aim of this article consists in proving that Fukui's conjecture holds for non-degenerate convenient weighted homogeneous polynomials, with no condition on the number of variables, up to the arc-analytic equivalence, a characterization of the blow-Nash equivalence.

It is proved in [2] that the arc-analytic type of a singular Brieskorn polynomial determines its exponents. The main idea consists in using a convolution formula in order to reduce to a combinatorial problem to prove that we can recover the exponents of such a polynomial from an arc-analytic invariant, namely the zeta function.

Notice that Brieskorn polynomials are non-degenerate convenient weighted homogeneous polynomials. However, since the variables are not separated, we can't use the convolution formula anymore to recover the weights of such a weighted homogeneous polynomial. Instead, the main idea of this article consists in using a formula allowing one to compute the zeta function of a non-degenerate polynomial from its Newton polyhedron, in order to reduce to a similar combinatorial problem as in the Brieskorn polynomials case.

The main theorem of this article states that the arc-analytic type of a singular non-degenerate convenient weighted homogeneous polynomial determines its weights.

**Theorem 1.1** (Main theorem). *Let  $f, g$  be two non-degenerate convenient weighted homogeneous polynomials. If  $f$  and  $g$  are arc-analytically equivalent, then either they are both non-singular or they share the same weights.*

We briefly recall the definitions of the objects involved in the previous statement.

**Remark 1.2.** In this article a *non-degenerate* polynomial is a polynomial which is non-degenerate with respect to its Newton polyhedron. See Definition 3.8.

**Definition 1.3.** A polynomial  $f \in \mathbb{R}[x_1, \dots, x_d]$  is said to be *convenient* if its Newton polyhedron  $\star$  intersects all the coordinate axes, i.e. for all  $i = 1, \dots, d$  a pure monomial of the form  $x_i^{\delta_i}$ ,  $\delta_i \in \mathbb{N}_{>0}$ , appears in the expansion of  $f$  with a nonzero coefficient.

**Definition 1.4.** A polynomial  $f \in \mathbb{R}[x_1, \dots, x_d]$  is said to be *weighted homogeneous* if there exists  $(w_1, \dots, w_d) \in \mathbb{N}_{>0}^d$  and  $w \in \mathbb{N}_{>0}$  such that

$$\forall \lambda \in \mathbb{R}, f(\lambda^{w_1} x_1, \dots, \lambda^{w_d} x_d) = \lambda^w f(x_1, \dots, x_d)$$

**Remark 1.5.** Under the assumptions of Theorem 1.1, the weight system  $(w_1, \dots, w_d) \in \mathbb{N}_{>0}^d$  of  $f$  is well defined up to a multiplicative factor. Notice also that reordering the variables doesn't change the arc-analytic type of a polynomial.

Hence, from now on, we refer to the weights of  $f$  as the unique primitive vector  $^\dagger (w_1, \dots, w_d) \in \mathbb{N}_{>0}^d$  such that  $f$  is weighted homogeneous with respect to the weights  $(w_1, \dots, w_d)$  and we assume that  $w_1 \geq \dots \geq w_d$ .

This article is structured as follows. First, Section 2 recalls the definition of the arc-analytic equivalence together with the definition and the needed properties of the invariant of the arc-analytic equivalence which we will use. Then, Section 3 introduces a formula to compute the above cited invariant for a non-degenerate polynomial. Finally, Section 4 contains the essence of the proof of Theorem 1.1.

**Acknowledgements.** I am very grateful to Toshizumi Fukui for his warm welcome in Saitama University and for our fruitful discussions. I would like to thank Adam Parusiński for his useful comments on a preliminary version of this work.

---

\* See Definition 3.1.

† i.e  $\gcd(w_1, \dots, w_d) = 1$ .

## 2 Recollection

### 2.1 The arc-analytic equivalence

The arc-analytic equivalence is a characterization of the blow-Nash equivalence of G. Fichou [8, 9, 10]. It avoids using Nash modifications in its definition and it allows one to prove, as expected, that it is an equivalence relation on Nash function germs.

G. Fichou [8, 9] proved that the blow-Nash equivalence admits no continuous moduli for isolated singularities. A. Parusiński and L. Paunescu [25] recently proved that the arc-analytic equivalence admits no continuous moduli even for families of non-isolated singularities.

**Definition 2.1** ([2, Definition 7.5]). Two Nash function germs  $\star f, g : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$  are *arc-analytically equivalent* if  $f = g \circ \varphi$  where

- $\varphi : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}^d, 0)$  is a semialgebraic homeomorphism,
- $\varphi$  is arc-analytic<sup>†</sup>,
- there exists  $c > 0$  such that  $|\det d\varphi| > c$  where  $d\varphi$  is defined.

**Remark 2.2** ([4, Theorem 3.5]). Let  $\varphi : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}^d, 0)$  be a semialgebraic homeomorphism which is also arc-analytic. If there exists  $c > 0$  such that  $|\det d\varphi| > c$  where  $d\varphi$  is defined then  $\varphi^{-1}$  is also arc-analytic and there exists  $c' > 0$  such that  $|\det d\varphi^{-1}| > c'$  where  $d\varphi^{-1}$  is defined

**Proposition 2.3** ([2, §7]). • *The arc-analytic equivalence is an equivalence relation.*

- *Two Nash function germs are arc-analytically equivalent if and only if they are blow-Nash equivalent.*

### 2.2 A motivic invariant of the arc-analytic equivalence

This section recalls the definition and some properties of the motivic zeta function introduced in [2]. We adopt the notations of the survey [3]. This motivic zeta function is an invariant of the arc-analytic equivalence whose construction is similar to the motivic zeta functions of Denef–Loeser [7] as the ones of Koike–Parusiński [17] and Fichou [8, 9].

**Proposition 2.4** ([24, §4.2]). *A semialgebraic subset  $S$  of  $\mathbb{P}_{\mathbb{R}}^n$  is an  $\mathcal{AS}$ -set if for every real analytic arc  $\gamma : (-1, 1) \rightarrow \mathbb{P}_{\mathbb{R}}^n$  satisfying  $\gamma((-1, 0)) \subset S$ , there exists  $\varepsilon > 0$  such that  $\gamma((0, \varepsilon)) \subset S$ .*

**Definition 2.5.** Let  $K_0(\mathcal{AS})$  be the free abelian group spanned by symbols  $\ddagger [X]$  with  $X \in \mathcal{AS}$  modulo the following relations

- (1) If there is a bijection  $X \rightarrow Y$  whose graph is  $\mathcal{AS}$  then  $[X] = [Y]$ .
- (2) For  $X \in \mathcal{AS}$  and  $Y \subset X$  be a closed  $\mathcal{AS}$ -subset we set  $[X \setminus Y] + [Y] = [X]$ .

The cartesian product induces a ring structure:

- (3)  $[X][Y] = [X \times Y]$ .

**Remark 2.6.** • We denote by  $0 = [\emptyset]$  the class of the empty set. It is the unit of the addition.

- We denote by  $1 = [\text{pt}]$  the class of the point. It is the unit of the multiplication.

- We denote by  $\mathbb{L}_{\mathcal{AS}} = [\mathbb{R}]$  the class of the affine line and we set  $\mathcal{M}_{\mathcal{AS}} = K_0(\mathcal{AS}) \left[ \mathbb{L}_{\mathcal{AS}}^{-1} \right]$ .

**Theorem 2.7** ([21],[8],[22]). *There is a unique map  $\beta : \mathcal{AS} \rightarrow \mathbb{Z}[u]$ , called the virtual Poincaré polynomial, such that*

- $\beta$  factorises through a ring morphism  $\beta : K_0(\mathcal{AS}) \rightarrow \mathbb{Z}[u]$ .
- If  $X \neq \emptyset$  then  $\deg \beta(X) = \dim X$  and the leading coefficient of  $\beta(X)$  is positive<sup>§</sup>.
- If  $X$  compact and non-singular then  $\beta(X) = \sum_i \dim H_i(X, \mathbb{Z}_2) u^i$ .

**Remark 2.8.** The virtual Poincaré polynomial induces a ring morphism  $\beta : \mathcal{M}_{\mathcal{AS}} \rightarrow \mathbb{Z}[u, u^{-1}]$ .

<sup>\*</sup> i.e. real analytic germs with semialgebraic graph.

<sup>†</sup> i.e.  $\varphi$  maps real analytic arcs to real analytic arcs by composition, it is a notion defined by K. Kurdyka [20].

<sup>‡</sup> It is well defined since  $\mathcal{AS}$  is a set.

<sup>§</sup>  $\beta(\emptyset) = 0$

The following Grothendieck group is an adaptation of the one of Guibert–Loeser–Merle [15] to our settings.

**Definition 2.9** ([2, Definition 3.4]). For  $n \in \mathbb{N}_{>0}$ , we denote by  $K_0(\mathcal{AS}_{\text{mon}}^n)$  the free abelian group spanned by symbols

$$[\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*]$$

where  $X \in \mathcal{AS}$ , the graph  $\Gamma_{\varphi_X} \in \mathcal{AS}$ , the graph of the action  $\Gamma_{\mathbb{R}^* \times X \rightarrow X} \in \mathcal{AS}$  and finally for all  $(\lambda, x) \in \mathbb{R}^* \times X$ ,  $\varphi_X(\lambda \cdot x) = \lambda^n \varphi_X(x)$  modulo the following relations

- (1) If there exists  $f : X \rightarrow Y$  a  $\mathbb{R}^*$ -equivariant bijection with  $\mathcal{AS}$ -graph such that

$$\begin{array}{ccc} X & \xrightarrow[f \simeq]{} & Y \\ \varphi_X \searrow & & \swarrow \varphi_Y \\ & \mathbb{R}^* & \end{array}$$

then we set

$$[\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*] = [\varphi_Y : \mathbb{R}^* \circ Y \rightarrow \mathbb{R}^*]$$

- (2) If  $Y$  is a  $\mathbb{R}^*$ -invariant closed  $\mathcal{AS}$ -subset of  $X$  then

$$[\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*] = [\varphi_{X|Y} : \mathbb{R}^* \circ Y \rightarrow \mathbb{R}^*] + [\varphi_{X|X \setminus Y} : \mathbb{R}^* \circ X \setminus Y \rightarrow \mathbb{R}^*]$$

- (3) Let  $\varphi_Y : \mathbb{R}^* \circ_\tau Y \rightarrow \mathbb{R}^*$  be a symbol and  $\psi = \varphi_Y \text{pr}_Y : Y \times \mathbb{R}^m \rightarrow \mathbb{R}^*$ . Let  $\sigma$  and  $\sigma'$  be two actions of  $\mathbb{R}^*$  on  $Y \times \mathbb{R}^m$  which are two liftings<sup>\*</sup> of  $\tau$  then  $\psi : \mathbb{R}^* \circ_\sigma (Y \times \mathbb{R}^m) \rightarrow \mathbb{R}^*$  and  $\psi : \mathbb{R}^* \circ_{\sigma'} (Y \times \mathbb{R}^m) \rightarrow \mathbb{R}^*$  are two symbols and we add the relation

$$[\psi : \mathbb{R}^* \circ_\sigma (Y \times \mathbb{R}^m) \rightarrow \mathbb{R}^*] = [\psi : \mathbb{R}^* \circ_{\sigma'} (Y \times \mathbb{R}^m) \rightarrow \mathbb{R}^*]$$

The fiber product over  $\mathbb{R}^*$  induces a ring structure:

- (4) We add the relation

$$[\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*] [\varphi_Y : \mathbb{R}^* \circ Y \rightarrow \mathbb{R}^*] = [X \times_{\mathbb{R}^*} Y \rightarrow \mathbb{R}^*]$$

where the action of  $\mathbb{R}^*$  on  $X \times_{\mathbb{R}^*} Y$  is diagonal from the previous ones.

The cartesian product induces a structure of  $K_0(\mathcal{AS})$ -algebra<sup>†</sup>:

- (5) Let  $[A] \in K_0(\mathcal{AS})$  and  $[\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*] \in K_0(\mathcal{AS}_{\text{mon}}^n)$  then we set

$$[A] \cdot [\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*] = [\varphi_X \text{pr}_X : A \times X \rightarrow \mathbb{R}^*]$$

where the action is trivial on  $A$ .

**Definition 2.10.** We set  $K_0(\mathcal{AS}_{\text{mon}}) = \varinjlim K_0(\mathcal{AS}_{\text{mon}}^n)$  where the direct system is induced by modifying the  $\mathbb{R}^*$ -action by  $\lambda \cdot_{K_0(\mathcal{AS}_{\text{mon}}^n)} x = \lambda^k \cdot_{K_0(\mathcal{AS}_{\text{mon}}^m)} x$  when  $n = km$ . It is a  $K_0(\mathcal{AS})$ -algebra.

**Remark 2.11.** • We denote by  $0 = [\emptyset]$  the class of the empty set. It is the unit of the addition.

• We set  $\mathbb{1} = [\text{id} : \mathbb{R}^* \rightarrow \mathbb{R}^*]$  where the action is defined by translation. It is the unit of the multiplication.

• We denote by  $\mathbb{L} = \mathbb{L}_{\mathcal{AS}} \cdot \mathbb{1}$  the class of the affine line and we set  $\mathcal{M} = K_0(\mathcal{AS}_{\text{mon}})[\mathbb{L}^{-1}]$ .  $\mathcal{M}$  has a natural structure of  $\mathcal{M}_{\mathcal{AS}}$ -algebra.

The following properties of  $K_0(\mathcal{AS}_{\text{mon}})$  and  $\mathcal{M}_{\mathcal{AS}}$  will be useful in what follows.

**Proposition 2.12** ([2, End of §3]). The map  $\mathcal{AS}_{\text{mon}}^n \rightarrow \mathcal{AS}$  defined by  $(\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*) \mapsto X$  induces:

- A morphism of  $K_0(\mathcal{AS})$ -modules  $\overline{\cdot} : K_0(\mathcal{AS}_{\text{mon}}) \rightarrow K_0(\mathcal{AS})$ ,
- A morphism of  $\mathcal{M}_{\mathcal{AS}}$ -modules  $\overline{\cdot} : \mathcal{M} \rightarrow \mathcal{M}_{\mathcal{AS}}$ .

<sup>\*</sup> i.e.  $\text{pr}_Y(\lambda \cdot_\sigma x) = \lambda \cdot_\tau \text{pr}_Y(x)$ .

<sup>†</sup> The algebra structure is given by the structural morphism  $K_0(\mathcal{AS}) \rightarrow K_0(\mathcal{AS}_{\text{mon}}^n)$  defined by  $[A] \mapsto [\text{pr}_{\mathbb{R}^*} : A \times \mathbb{R}^* \rightarrow \mathbb{R}^*]$  where the  $\mathbb{R}^*$ -action is  $\lambda \cdot (a, r) = (a, \lambda^n r)$ .

**Proposition 2.13** ([2, Proposition 4.16]). *Let  $\varepsilon \in \{+, -\}$ . The map  $\mathcal{AS}_{\text{mon}}^n \rightarrow \mathcal{AS}$  defined by  $(\varphi_X : \mathbb{R}^* \circ X \rightarrow \mathbb{R}^*) \mapsto \varphi_X^{-1}(\varepsilon 1)$  induces:*

- *A morphism of  $K_0(\mathcal{AS})$ -algebras  $F^\varepsilon : K_0(\mathcal{AS}_{\text{mon}}) \rightarrow K_0(\mathcal{AS})$ ,*
- *A morphism of  $\mathcal{M}_{\mathcal{AS}}$ -algebras  $F^\varepsilon : \mathcal{M} \rightarrow \mathcal{M}_{\mathcal{AS}}$ .*

**Remark 2.14.** The morphisms of the last proposition are compatible with the products since the fiber product over a point coincides with the cartesian product.

The following zeta function is a real analog of the equivariant motivic zeta function of Denef–Loeser.

**Definition 2.15** ([2, Definition 4.2]). Let  $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$  be a Nash function germ. We set

$$Z_f(T) = \sum_{n \geq 1} \left[ \text{ac}_f^n : \mathbb{R}^* \circ \mathfrak{X}_n(f) \rightarrow \mathbb{R}^* \right] \mathbb{L}^{-nd} T^n \in \mathcal{M}[[T]]$$

where  $\mathfrak{X}_n(f) = \{\gamma = a_1 t + \dots + a_n t^n, a_i \in \mathbb{R}^d, f\gamma(t) = ct^n + \dots, c \neq 0\}$ ,  $\text{ac}_f^n : \mathfrak{X}_n(f) \rightarrow \mathbb{R}^*$  is the angular component map defined by  $\text{ac}_f^n(\gamma) = \text{ac}(f\gamma) = c$  and where the action is defined by  $\lambda \cdot \gamma(t) = \gamma(\lambda t)$ .

**Theorem 2.16** ([2, Theorem 7.11]). *If  $f, g : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$  are two arc-analytically equivalent Nash function germs then  $Z_f(T) = Z_g(T)$ .*

**Definition 2.17** ([2, Definition 6.6]). Let  $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$  be a Nash function germ. We define the modified zeta function of  $f$  by

$$\tilde{Z}_f(T) = Z_f(T) - \frac{\mathbb{1} - Z_f^{\text{naive}}(T)}{\mathbb{1} - T} + \mathbb{1}$$

where  $Z_f^{\text{naive}}(T)$  is defined by applying  $\alpha \mapsto \bar{\alpha} \cdot \mathbb{1}$  to the coefficients of  $Z_f(T)$ .

**Remark 2.18.** Particularly, the modified zeta function is an invariant of the arc-analytic equivalence too. Moreover, it encodes the same information as  $Z_f(T)$  ([2, Corollary 6.14]):

$$Z_f(T) = \tilde{Z}_f(T) + \frac{\mathbb{1} - \mathbb{L}^{-1} \tilde{Z}_f^{\text{naive}}(T)}{\mathbb{1} - \mathbb{L}^{-1} T} - \mathbb{1}$$

### 3 The modified zeta function of a non-degenerate polynomial

**Definition 3.1.** Let  $f = \sum_{v \in \mathbb{N}^d} c_v x^v \in \mathbb{R}[x_1, \dots, x_d]$ . We define the Newton polyhedron of  $f$  by

$$\Gamma_f = \text{Conv} \left( \bigcup_{\substack{v \in \mathbb{N}^d \\ c_v \neq 0}} (c_v + \mathbb{R}_{\geq 0}^d) \right)$$

**Definition 3.2.** For  $\tau$  a face of  $\Gamma_f$ , we set  $f_\tau(x) = \sum_{c_v \in \tau} c_v x^v$ .

**Definition 3.3.** We define the supporting function  $m : \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}_{\geq 0}$  by  $m(k) = \inf \{k \cdot x, x \in \Gamma_f\}$ .

**Definition 3.4.** The trace of  $k \in \mathbb{R}_{\geq 0}^d$  is  $\tau(k) = \{x \in \Gamma_f, k \cdot x = m(k)\}$ .

**Definition 3.5.** We define the dual cone of a face  $\tau$  of  $\Gamma_f$  by  $\sigma(\tau) = \{k \in \mathbb{R}_{\geq 0}^d, \tau(k) = \tau\}$ .

**Notation 3.6.** For a face  $\tau$  of  $\Gamma_f$ , we set  $\tilde{\sigma}(\tau) = \sigma(\tau) \cap \mathbb{N}^d$ .

**Notation 3.7.** We denote by  $\Gamma_f^c$  the set of compact faces of  $\Gamma_f$ .

**Definition 3.8.** A polynomial  $f \in \mathbb{R}[x_1, \dots, x_d]$  is non-degenerate (with respect to its Newton polyhedron) if

$$\forall \tau \in \Gamma_f^c, \left\{ x \in (\mathbb{R}^*)^d, \forall i = 1, \dots, d, \frac{\partial f_\tau}{\partial x_i}(x) = 0 \right\} = \emptyset$$

**Remark 3.9.** For a face  $\tau \in \Gamma_f^c$ , notice that  $f_\tau$  is weighted homogeneous and hence, by Euler formula,

$$\left\{ x \in (\mathbb{R}^*)^d, \frac{\partial f_\tau}{\partial x_1}(x) = \dots = \frac{\partial f_\tau}{\partial x_d}(x) = 0 \right\} = \left\{ x \in (\mathbb{R}^*)^d, f_\tau(x) = \frac{\partial f_\tau}{\partial x_1}(x) = \dots = \frac{\partial f_\tau}{\partial x_d}(x) = 0 \right\}$$

**Lemma 3.10** ([26, Proposition 3.13]). For  $\tau \in \Gamma_f^c$  and  $k \in \tilde{\sigma}(\tau)$ , we set

$$\left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] = \left[ f_\tau : \mathbb{R}^* \curvearrowright \left( (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right) \rightarrow \mathbb{R}^* \right] \in K_0(\mathcal{AS}_{\text{mon}}^{m(k)})$$

where the action is given by  $\lambda \cdot (x_i)_i = (\lambda^{k_i} x_i)_i$ .

Then  $\left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] \in K_0(\mathcal{AS}_{\text{mon}})$  doesn't depend on the choice of  $k \in \tilde{\sigma}(\tau)$ .

**Notation 3.11.** For  $m \in \mathbb{Z}$ , we denote by  $\mathcal{F}^m \mathcal{M}$  the subgroup of  $\mathcal{M}$  spanned by the elements of the form  $[S] \mathbb{L}^{-i}$  where  $i - \dim S \geq m$ . It defines a filtration and we denote by  $\widehat{\mathcal{M}}$  the completion of  $\mathcal{M}$  with respect to this filtration.

**Remark 3.12.** Notice that  $\beta^\cdot$  and  $\beta F^\pm$  factorises through the image of  $\mathcal{M} \rightarrow \widehat{\mathcal{M}}$  since the kernel is  $\cap \mathcal{F}^m \mathcal{M}$  and if  $\alpha \in \cap \mathcal{F}^m \mathcal{M}$  then  $\beta(\bar{\alpha}) = 0$  and  $\beta F^\varepsilon(\alpha) = 0$  by consideration on the degree.

A similar formula of the one of the following result has been proved in different settings: by A. N. Varchenko [29] for the zeta function of the monodromy, by J. Denef and F. Loeser [6, §5] for the topological zeta function, by J. Denef and K. Hoornaert [5, Theorem 4.2] for the Igusa  $p$ -adic zeta function and by G. Guibert for the motivic zeta function [14, Proposition 2.1.3]. The proof of G. Fichou and T. Fukui [12] already relies on an adaptation of this construction to the virtual Poincaré polynomial.

**Theorem 3.13** ([2, Theorem 5.15]). Let  $f \in \mathbb{R}[x_1, \dots, x_d]$  be non-degenerate, then

$$Z_f(T) = \sum_{\tau \in \Gamma_f^c} \left( \left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] + \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \frac{\mathbb{L}^{-1}T}{1 - \mathbb{L}^{-1}T} \right) S_\tau \in \widehat{\mathcal{M}}[[T]]$$

where  $\left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \in K_0(\mathcal{AS})$  and  $S_\tau = \sum_{k \in \tilde{\sigma}(\tau)} \mathbb{L}^{-|k|} T^{m(k)}$  with  $|k| = \sum_{i=1}^d k_i$ .

**Remark 3.14.** Notice that for the purpose of the previous theorem, we may have simply set  $\widehat{\mathcal{M}} = \mathcal{M} \left[ \frac{1}{1 - \mathbb{L}^a}, a \in \mathbb{Z} \right]$ .

The proof of the Main Theorem relies on the induced formula for the modified zeta function.

**Corollary 3.15.** Let  $f$  be non-degenerate, then

$$\tilde{Z}_f(T) = \sum_{\tau \in \Gamma_f^c} \left( \left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] - \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot \mathbb{1} + \frac{(\mathbb{L} - \mathbb{1})^d}{1 - T} \right) S_\tau - \frac{T}{1 - T} \in \widehat{\mathcal{M}}[[T]]$$

*Proof.*

$$\begin{aligned} Z_f^{\text{naive}} &= \sum_{\tau \in \Gamma_f^c} \left( \left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] \cdot \mathbb{1} + \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot (\mathbb{L} - \mathbb{1}) \frac{\mathbb{L}^{-1}T}{1 - \mathbb{L}^{-1}T} \right) S_\tau \\ &= \sum_{\tau \in \Gamma_f^c} \left( (\mathbb{L} - \mathbb{1})^d - \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot \mathbb{1} + \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot (\mathbb{L} - \mathbb{1}) \frac{\mathbb{L}^{-1}T}{1 - \mathbb{L}^{-1}T} \right) S_\tau \\ &= \sum_{\tau \in \Gamma_f^c} \left( (\mathbb{L} - \mathbb{1})^d + \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot \mathbb{1} \frac{T - \mathbb{1}}{1 - \mathbb{L}^{-1}T} \right) S_\tau \end{aligned}$$

Thus

$$\begin{aligned}
\tilde{Z}_f(T) &= Z_f(T) - \frac{Z_f^{\text{naive}}(T) - \mathbb{1}}{T - \mathbb{1}} + \mathbb{1} \\
&= \sum_{\tau \in \Gamma_f^c} \left( \left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] + \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \frac{\mathbb{L}^{-1}T}{\mathbb{1} - \mathbb{L}^{-1}T} + \frac{(\mathbb{L} - \mathbb{1})^d}{\mathbb{1} - T} - \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot \mathbb{1} \frac{\mathbb{1}}{\mathbb{1} - \mathbb{L}^{-1}T} \right) S_\tau \\
&\quad - \frac{T}{\mathbb{1} - T} \\
&= \sum_{\tau \in \Gamma_f^c} \left( \left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] - \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot \mathbb{1} + \frac{(\mathbb{L} - \mathbb{1})^d}{\mathbb{1} - T} \right) S_\tau - \frac{T}{\mathbb{1} - T}
\end{aligned}$$

■

## 4 Application to convenient non-degenerate weighted homogeneous polynomials

Throughout this section, we use the notation  $\tilde{Z}_f(T) = A_f(T) + B_f(T) - \frac{T}{\mathbb{1} - T}$  where

$$A_f(T) = \sum_{\tau \in \Gamma_f^c} \left( \left[ (\mathbb{R}^*)^d \setminus f_\tau^{-1}(0) \right] - \left[ f_\tau^{-1}(0) \cap (\mathbb{R}^*)^d \right] \cdot \mathbb{1} \right) S_\tau$$

and

$$B_f(T) = \frac{(\mathbb{L} - \mathbb{1})^d}{\mathbb{1} - T} \sum_{\tau \in \Gamma_f^c} S_\tau$$

We will see that the modified zeta function of a convenient non-degenerate weighted homogeneous polynomials is very similar to the one obtained for a Brieskorn polynomial [2, §8] using the convolution formula.

### 4.1 Computation of $B_f(T)$

**Lemma 4.1.** *Let  $f \in \mathbb{R}[x_1, \dots, x_d]$  be a convenient non-degenerate weighted homogeneous polynomial. We denote by  $\delta_i$  the exponent of the pure monomial  $x_i$  of  $f$ . Then*

$$B_f(T) - \frac{T}{\mathbb{1} - T} = - \sum_{m \geq 1} \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{m}{\delta_i} \rfloor} T^m$$

*Proof.* Under our assumptions  $m(k) = k \cdot (0, \dots, \delta_i, \dots, 0)$  for some  $i$ , thus the coefficient of degree  $m$  is of the form:

$$\begin{aligned}
(\mathbb{L} - \mathbb{1})^d \sum_{\substack{k \in (\mathbb{N} \setminus \{0\})^d \\ m(k) \leq m}} \mathbb{L}^{-|k|} - \mathbb{1} &= (\mathbb{L} - \mathbb{1})^d \sum_{k \in (\mathbb{N} \setminus \{0\})^d} \mathbb{L}^{-|k|} - (\mathbb{L} - \mathbb{1})^d \sum_{\substack{k \in (\mathbb{N} \setminus \{0\})^d \\ m(k) > m}} \mathbb{L}^{-|k|} - \mathbb{1} \\
&= \mathbb{1} - (\mathbb{L} - \mathbb{1})^d \prod_{i=1}^d \sum_{k_i \geq \lfloor \frac{m}{\delta_i} \rfloor + 1} \mathbb{L}^{-k_i} - \mathbb{1} \\
&= -(\mathbb{L} - \mathbb{1})^d \prod_{i=1}^d \frac{\mathbb{L}^{-\lfloor \frac{m}{\delta_i} \rfloor - 1}}{\mathbb{1} - \mathbb{L}^{-1}} \\
&= -(\mathbb{L} - \mathbb{1})^d \frac{\prod_{i=1}^d \mathbb{L}^{-\lfloor \frac{m}{\delta_i} \rfloor}}{(\mathbb{L} - \mathbb{1})^d}
\end{aligned}$$

■

**Corollary 4.2.** *Let  $f \in \mathbb{R}[x_1, \dots, x_d]$  be a convenient non-degenerate weighted homogeneous polynomial. We denote by  $\delta_i$  the exponent of the pure monomial  $x_i$  of  $f$ . Denote by  $a_n$  the coefficients of  $\tilde{Z}_f(T)$ , so that*

$$\tilde{Z}_f(T) = \sum_{n \geq 1} a_n T^n$$

*Let  $m \in \mathbb{N}_{>0}$  such that  $\forall i, \delta_i \nmid m$ . Then*

$$a_m = -\mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{m}{\delta_i} \rfloor}$$

*Proof.* Under our assumptions  $m(k) = k \cdot (0, \dots, \delta_i, \dots, 0)$  for some  $i$ . Since  $\forall i, \delta_i \nmid m$  we get that  $A_f(T)$  doesn't contribute to the coefficient of degree  $m$ . ■

## 4.2 Computation of $A_f(T)$

**Remark 4.3.** When  $f$  is a convenient non-degenerate weighted homogeneous polynomial, the cones dual to the faces of its Newton polyhedron are simplicial. More precisely such a cone is spanned by  $w$  and at most  $d-1$  coordinate vectors, where  $w$  is the primitive vector normal to the compact facet.

**Notation 4.4.** For  $I \subset \{1, \dots, d\}$  we set  $\delta_I = \text{lcm}\{\delta_i, i \in I\}$ .

**Remark 4.5.** Obviously  $w$  is the weight vector

$$w = \left( \frac{\delta_{\{1, \dots, d\}}}{\delta_i} \right)_{i=1, \dots, d}$$

so that

$$\mathbb{L}^{-|w|} T^{m(w)} = \mathbb{L}^{-\sum_{i=1}^d \frac{\delta_{\{1, \dots, d\}}}{\delta_i}} T^{\delta_{\{1, \dots, d\}}}$$

**Lemma 4.6.** *Let  $f \in \mathbb{R}[x_1, \dots, x_d]$  be a convenient non-degenerate weighted homogeneous polynomial. For  $\emptyset \neq I \subset \{1, \dots, d\}$  we denote by  $\tau_I$  the face of  $\Gamma_f^c$  intersecting the  $x_i$ -axis for  $i \in I$ . Then*

$$S_{\tau_I} = \frac{1}{(\mathbb{L} - 1)^{d-|I|}} \sum_{r \geq 1} \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{r\delta_I}{\delta_i} \rfloor} T^{r\delta_I}$$

*Proof.* Notice that  $\sigma_I$ , the dual cone of  $\tau_I$ , is the cone spanned by  $w$  and  $e_i$  for  $i \notin I$ .

Denote by  $a_m$  the coefficients of  $S_{\tau_I}$  so that  $S_{\tau_I} = \sum_{m \geq 1} a_m T^m$ .

Fix  $m \in \mathbb{N} \setminus \{0\}$ . We may assume that  $\delta_I \mid m$  otherwise  $a_m = 0$ .

Let  $k \in (\mathbb{N} \setminus \{0\})^d$ . Then  $k \in \sigma_I$  and satisfies  $m(k) = m$  if and only if for  $i \in I$ ,  $k_i \delta_i = m$  and for  $j \notin I$ ,  $k_j \geq \lfloor \frac{m}{\delta_j} \rfloor + 1$ .

Thus

$$\begin{aligned} a_m &= \sum_{\substack{k_j \geq \lfloor \frac{m}{\delta_j} \rfloor + 1 \\ j \notin I}} \mathbb{L}^{-|k|} \\ &= \mathbb{L}^{-\sum_{i \in I} \frac{m}{\delta_i}} \sum_{\substack{k_j \geq \lfloor \frac{m}{\delta_j} \rfloor + 1 \\ j \notin I}} \mathbb{L}^{-k_j} \\ &= \mathbb{L}^{-\sum_{i \in I} \frac{m}{\delta_i}} \prod_{j \notin I} \frac{\mathbb{L}^{-\lfloor \frac{m}{\delta_j} \rfloor - 1}}{1 - \mathbb{L}^{-1}} \\ &= \frac{\mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{m}{\delta_i} \rfloor}}{(\mathbb{L} - 1)^{d-|I|}} \end{aligned}$$

■



**Corollary 4.7.**

$$A_f(T) = \sum_{\emptyset \neq I \subset \{1, \dots, d\}} \left( \left[ (\mathbb{R}^*)^{|I|} \setminus f_{\tau_I}^{-1}(0) \right] - \left[ f_{\tau_I}^{-1}(0) \cap (\mathbb{R}^*)^{|I|} \right] \cdot \mathbb{1} \right) \sum_{r \geq 1} \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{r\delta_i}{\delta_I} \rfloor} T^{r\delta_I}$$

**Remark 4.8.** The formulae obtained in this section allow us to give a rationality formula for  $\tilde{Z}_f(T)$ . Indeed, for  $\emptyset \neq I \subset \{1, \dots, d\}$  and by writing the Euclidean division of  $r$  by  $\frac{\delta_{\{1, \dots, d\}}}{\delta_I}$ , we get

$$\begin{aligned} \sum_{r \geq 1} \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{r\delta_i}{\delta_I} \rfloor} T^{r\delta_I} &= \sum_{q \geq 0} \sum_{s=0}^{\frac{\delta_{\{1, \dots, d\}}}{\delta_I} - 1} \mathbb{L}^{-\sum_{i=1}^d \left( q \frac{\delta_{\{1, \dots, d\}}}{\delta_i} + \lfloor \frac{s\delta_i}{\delta_I} \rfloor \right)} T^{q\delta_{\{1, \dots, d\}} + s\delta_I} - \mathbb{1} \\ &= \frac{\sum_{s=0}^{\frac{\delta_{\{1, \dots, d\}}}{\delta_I} - 1} \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{s\delta_i}{\delta_I} \rfloor} T^{s\delta_I}}{\mathbb{1} - \mathbb{L}^{-\sum_{i=1}^d \frac{\delta_{\{1, \dots, d\}}}{\delta_i}} T^{\delta_{\{1, \dots, d\}}}} - \mathbb{1} \\ &= \frac{\sum_{s=0}^{\frac{\delta_{\{1, \dots, d\}}}{\delta_I} - 1} \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{s\delta_i}{\delta_I} \rfloor} T^{s\delta_I}}{\mathbb{1} - \mathbb{L}^{-|w|} T^{m(w)}} - \mathbb{1} \end{aligned}$$

We can do the same for the formula of Lemma 4.1 by taking the Euclidean division of  $m$  by  $\delta_{\{1, \dots, d\}}$ .

### 4.3 Proof of the invariance of the weights

**Lemma 4.9.** Let  $f \in \mathbb{R}[x_1, \dots, x_d]$  be a convenient non-degenerate weighted homogeneous polynomial. Denote by  $a_n$  the coefficients of  $\tilde{Z}_f(T)$ , so that

$$\tilde{Z}_f(T) = \sum_{n \geq 1} a_n T^n$$

Denote by  $\delta_i$  the exponent of the pure monomial  $x_i$  of  $f$ . Assume that  $\delta_i \geq 2$ . Let  $m$  be such that  $2|m$  and  $\delta_i \nmid m$  for  $\delta_i \neq 2$ . Then

$$a_m = \alpha_2 \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{m}{\delta_i} \rfloor}$$

with  $\beta F^+(\alpha_2) \neq 0$ .

*Proof.* Let  $I_2 = \{i, \delta_i = 2\}$  then, by Lemma 4.1 and Corollary 4.7,

$$a_m = \left( \sum_{\emptyset \neq I \subset I_2} \left( \left[ (\mathbb{R}^*)^{|I|} \setminus f_{\tau_I}^{-1}(0) \right] - \left[ f_{\tau_I}^{-1}(0) \cap (\mathbb{R}^*)^{|I|} \right] \cdot \mathbb{1} \right) - \mathbb{1} \right) \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{m}{\delta_i} \rfloor}$$

Let

$$\alpha_2 = \sum_{\emptyset \neq I \subset I_2} \left( \left[ (\mathbb{R}^*)^{|I|} \setminus f_{\tau_I}^{-1}(0) \right] - \left[ f_{\tau_I}^{-1}(0) \cap (\mathbb{R}^*)^{|I|} \right] \cdot \mathbb{1} \right) - \mathbb{1}$$

Then

$$\begin{aligned} \beta F^+(\alpha_2) &= \beta F^+ \left( \sum_{\emptyset \neq I \subset I_2} \left( \left[ (\mathbb{R}^*)^{|I|} \setminus f_{\tau_I}^{-1}(0) \right] - \left[ f_{\tau_I}^{-1}(0) \cap (\mathbb{R}^*)^{|I|} \right] \cdot \mathbb{1} \right) - \mathbb{1} \right) \\ &= \sum_{\emptyset \neq I \subset I_2} \left( \beta \left( f_{\tau_I}^{-1}(1) \subset (\mathbb{R}^*)^{|I|} \right) - \beta \left( f_{\tau_I}^{-1}(0) \subset (\mathbb{R}^*)^{|I|} \right) \right) - 1 \\ &= \beta \left( f_{\tau_{I_2}}^{-1}(1) \subset (\mathbb{R})^{|I_2|} \right) - \beta \left( f_{\tau_{I_2}}^{-1}(0) \subset (\mathbb{R})^{|I_2|} \right) \end{aligned}$$

By a change of coordinates, we may assume that  $f_{\tau_{I_2}}$  is a homogeneous Brieskorn polynomial of degree 2. When we take the value at  $u = -1$ , which is just the Euler characteristic with compact support, the first term is odd by [10, Proposition 2.1] or [11, Proposition 2.1] whereas the second term is even by [10, Corollary 2.5] or [11, Proposition 2.1]. Hence  $\beta F^+(\alpha_2)(u = -1) \neq 0$  and so  $\beta F^+(\alpha_2) \neq 0$ .  $\blacksquare$

**Lemma 4.10.** *Let  $f \in \mathbb{R}[x_1, \dots, x_d]$  be a convenient non-degenerate weighted homogeneous polynomial. Denote by  $a_n$  the coefficients of  $\tilde{Z}_f(T)$ , so that*

$$\tilde{Z}_f(T) = \sum_{n \geq 1} a_n T^n$$

*Denote by  $\delta_i$  the exponent of the pure monomial  $x_i$  of  $f$ . Assume that  $\delta_i \geq 2$ . Let  $m$  be such that  $4|m$  and  $\delta_i \nmid m$  for  $\delta_i \neq 2, 4$ . Then*

$$a_m = \alpha_4 \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{m}{\delta_i} \rfloor}$$

*with  $\beta F^+(\alpha_4) \neq 0$ .*

*Proof.* Let  $I_4 = \{i, \delta_i | 4\}$  then, by Lemma 4.1 and Corollary 4.7,

$$a_m = \left( \sum_{\emptyset \neq I \subset I_4} \left( \left[ (\mathbb{R}^*)^{|I|} \setminus f_{\tau_I}^{-1}(0) \right] - \left[ f_{\tau_I}^{-1}(0) \cap (\mathbb{R}^*)^{|I|} \right] \cdot \mathbb{1} \right) - \mathbb{1} \right) \mathbb{L}^{-\sum_{i=1}^d \lfloor \frac{m}{\delta_i} \rfloor}$$

Let

$$\alpha_4 = \sum_{\emptyset \neq I \subset I_4} \left( \left[ (\mathbb{R}^*)^{|I|} \setminus f_{\tau_I}^{-1}(0) \right] - \left[ f_{\tau_I}^{-1}(0) \cap (\mathbb{R}^*)^{|I|} \right] \cdot \mathbb{1} \right) - \mathbb{1}$$

Then

$$\begin{aligned} \chi_c F^+(\alpha_4) &= \chi_c F^+ \left( \sum_{\emptyset \neq I \subset I_4} \left( \left[ (\mathbb{R}^*)^{|I|} \setminus f_{\tau_I}^{-1}(0) \right] - \left[ f_{\tau_I}^{-1}(0) \cap (\mathbb{R}^*)^{|I|} \right] \cdot \mathbb{1} \right) - \mathbb{1} \right) \\ &= \sum_{\emptyset \neq I \subset I_4} \left( \chi_c \left( f_{\tau_I}^{-1}(1) \subset (\mathbb{R}^*)^{|I|} \right) - \chi_c \left( f_{\tau_I}^{-1}(0) \subset (\mathbb{R}^*)^{|I|} \right) \right) - 1 \\ &= \sum_{\emptyset \neq I \subset I_4} \chi_c \left( f_{\tau_I}^{-1}(1) \subset (\mathbb{R}^*)^{|I|} \right) - \sum_{\emptyset \neq I \subset I_4} \chi_c \left( f_{\tau_I}^{-1}(0) \subset (\mathbb{R}^*)^{|I|} \right) - \chi_c(\{0\}) \\ &= \chi_c \left( f_{\tau_{I_4}}^{-1}(1) \subset (\mathbb{R})^{|I_4|} \right) - \chi_c \left( f_{\tau_{I_4}}^{-1}(0) \subset (\mathbb{R})^{|I_4|} \right) \end{aligned}$$

The last equality comes from the additivity of  $\chi_c$ . Indeed, for  $I \subset I_4$ , we have

$$\chi_c \left( f_{\tau_I}^{-1}(1) \subset (\mathbb{R}^*)^{|I|} \right) = \chi_c \left( f_{\tau_{I_4}}^{-1}(1) \subset (\mathbb{R})^{|I_4|}, \forall i \in I, x_i \neq 0, \forall j \in I_4 \setminus I, x_j = 0 \right)$$

We use the exact same argument for the second sum with the additional term  $\chi_c(\{0\})$  since  $0 \in f_{\tau_{I_4}}^{-1}(0)$ .

Denote by  $f_{\tau_{I_4}}^{\mathbb{C}}$  the complexification of  $f_{\tau_{I_4}}$ . Since

$$\chi_c \left( f_{\tau_{I_4}}^{-1}(x) \right) \equiv \chi \left( f_{\tau_{I_4}}^{\mathbb{C}-1}(x) \right) \pmod{2}$$

we have

$$\chi_c \left( f_{\tau_{I_4}}^{-1}(1) \subset (\mathbb{R})^{|I_4|} \right) - \chi_c \left( f_{\tau_{I_4}}^{-1}(0) \subset (\mathbb{R})^{|I_4|} \right) \equiv \chi \left( f_{\tau_{I_4}}^{\mathbb{C}-1}(1) \subset (\mathbb{C})^{|I_4|} \right) - \chi \left( f_{\tau_{I_4}}^{\mathbb{C}-1}(0) \subset (\mathbb{C})^{|I_4|} \right) \pmod{2}$$

Since  $f_{\tau_{I_4}}^{\mathbb{C}}$  is weighted homogeneous,  $f_{\tau_{I_4}}^{\mathbb{C}-1}(1)$  is homeomorphic to the Milnor fiber of  $f_{\tau_{I_4}}^{\mathbb{C}}$  at 0 [23, Lemma 9.4] and since  $f_{\tau_{I_4}}^{\mathbb{C}}$  is convenient and non-degenerate, the Euler characteristic of its Milnor fiber may be computed from its Newton polyhedron (See [23, p. 64] and [18, 1.10 Théorème I.(ii).]).

Since  $f_{\tau_{I_4}}^{\mathbb{C}}$  is non-degenerate,  $\chi_c \left( f_{\tau_{I_4}}^{\mathbb{C}-1}(0) \subset (\mathbb{C})^{|I_4|} \right)$  may be computed from its Newton polyhedron formula [16, Theorem 2].

Hence

$$\chi_c \left( f_{\tau_{I_4}}^{-1}(1) \subset (\mathbb{R})^{|I_4|} \right) - \chi_c \left( f_{\tau_{I_4}}^{-1}(0) \subset (\mathbb{R})^{|I_4|} \right) \pmod{2}$$

only depends on the Newton polyhedron of  $f_{\tau_{I_4}}$ .

So we may assume that

$$f_{\tau_{I_4}}(x) = \sum_i \varepsilon_i x_i^2 + \sum_j \eta_j x_j^4$$

with  $\varepsilon_i, \eta_j \in \{\pm 1\}$ . But then (see the proof of [2, Proposition 8.3])

$$\chi_c \left( f_{\tau_{I_4}}^{-1}(1) \subset (\mathbb{R})^{|I_4|} \right) - \chi_c \left( f_{\tau_{I_4}}^{-1}(0) \subset (\mathbb{R})^{|I_4|} \right) \equiv 1 \pmod{2}$$

Then  $\beta F^+(\alpha_4)(u = -1) = \chi_c F^+(\alpha_4) \neq 0$  and thus  $\beta F^+(\alpha_4) \neq 0$ . ■

**Remark 4.11.** The proof of Lemma 4.10 works as it is to prove Lemma 4.9. However the above proof of Lemma 4.9 is easier.

The previous results allow one to prove Theorem 1.1 using the proof of [2, Proposition 8.3]. We recall it for ease of reading.

The proof is structured as follows. We first handle the non-singular case so that we can focus on the singular one. In the latter, the idea is to prove that we can recover the weights of  $f$  a singular convenient non-degenerate weighted homogeneous polynomial from its modified zeta function. Since it is an invariant of the arc-analytic equivalence, this induces that the arc-analytic type of such a polynomial determines its weights.

More precisely we are going to deduce the exponents of the pure monomials of  $f$  from its modified zeta function, which is enough by Remark 4.5. The first step is to find an upper bound of these exponents in order to reduce to a finite number of possibilities. Finally, we construct (and solve) a linear system which determines these exponents.

*Proof of Theorem 1.1.* We first fix some notations.

Let  $f \in \mathbb{R}[x_1, \dots, x_d]$  be a convenient non-degenerate weighted homogeneous polynomial. Denote by  $a_n$  the coefficients of  $\tilde{Z}_f(T)$ , so that

$$\tilde{Z}_f(T) = \sum_{n \geq 1} a_n T^n$$

Denote by  $\delta_i$  the exponent of the pure monomial  $x_i$  of  $f$ .

Notice that  $f$  is singular if and only if  $\forall i, \delta_i \geq 2$ . Indeed, if, for example,  $\delta_1 = 1$  then  $f(x_1, \dots, x_d) = x_1 + h(x_2, \dots, x_d)$  and  $f$  is non-singular. Otherwise, if  $\forall i, \delta_i \geq 2$  then we may apply the Euler formula to the partial derivatives of  $f$  to show that  $f$  is singular at the origin.

If  $f$  is non-singular, then, by a change of coordinates, we may assume  $\delta_1 = 1$  and  $f$  is arc-analytically equivalent to  $(x_1, \dots, x_d) \mapsto x_1$  by applying the Nash inverse function theorem to  $(x_1, \dots, x_d) \mapsto (f(x_1, \dots, x_d), x_2, \dots, x_d)$ . Notice that in this case  $\tilde{Z}_f(T) = 0$ . Actually, by Corollary 4.2,  $f$  is non-singular if and only if  $\tilde{Z}_f(T) = 0$ .

Now let  $f$  and  $g$  be two convenient non-degenerate weighted homogeneous polynomials which are arc-analytically equivalent. If  $f$  is non-singular then  $\tilde{Z}_f(T) = \tilde{Z}_g(T) = 0$  and  $g$  is also non-singular. Moreover, they are both arc-analytically equivalent to  $(x_1, \dots, x_d) \mapsto x_1$ .

Hence, from now on, we may assume that  $\forall i \in \{1, \dots, d\}, \delta_i \geq 2$ .

Then, by Corollary 4.2,

$$\lim_p \frac{1 - \deg \beta(\overline{a_p})}{p} = \lim_p \frac{\sum_{i=1}^d \left\lfloor \frac{p}{\delta_i} \right\rfloor}{p} = \sum_{i=1}^d \frac{1}{\delta_i}$$

where  $p$  goes through the prime numbers.

Hence, we may deduce from [2, Lemma 8.5] an upper bound  $K$  of the set  $\{\delta_1, \dots, \delta_d\}$ .

For  $q \in \mathbb{N}$ , set  $\text{mult}(q) = \#\{i, \delta_i = q\}$ . The goal is to prove that we can retrieve  $\text{mult}(q)$  for  $q \leq K$  from  $\tilde{Z}_f(T)$ . This will prove, using Remark 4.5, that we may recover the weights of  $f$  from  $\tilde{Z}_f(T)$ . Which is enough to achieve the proof of Theorem 1.1 since  $\tilde{Z}_f(T)$  is an invariant

of the arc-analytic equivalence.

First, to lighten the redaction, we enlarge the set  $\{1, \dots, K\}$  as follows. Set

$$\mathcal{P} = \{p \text{ prime}, p \leq K\}$$

and for  $p \in \mathcal{P}$ , set

$$\gamma_p = \max\{\gamma, p^\gamma \leq K\}$$

Set

$$Q = \left\{ \prod_{p \in \mathcal{P}} p^{\alpha_p}, 0 \leq \alpha_p \leq \gamma_p \right\}$$

Then  $\{\delta_1, \dots, \delta_d\} \subset \{1, \dots, K\} \subset Q$  and we are going to compute  $\text{mult}(q)$  for  $q \in Q$  from the coefficients of  $\tilde{Z}_f(T)$ .

Notice that  $\text{mult } 1 = 0$ .

The proof is now divided into different steps.

Step 1: Equations involving  $\text{mult}(q)$  for  $q \in Q$  such that  $6|q$ .

Let  $q = \prod_{p \in \mathcal{P}} p^{\alpha_p}$  with  $\alpha_2, \alpha_3 \geq 1$  and  $0 \leq \alpha_p \leq \gamma_p$  for  $p \in \mathcal{P} \setminus \{2, 3\}$ . By the Chinese Remainder Theorem, we may find  $n \in \mathbb{N} \setminus \{0, 1\}$  such that 1 is the only element of  $Q$  dividing  $n-1$  or  $n+1$  and such that the only elements of  $Q$  dividing  $n$  are the divisors of  $q$ .

Indeed, it suffices to choose  $n$  such that

$$\begin{cases} n \equiv p^{\alpha_p} \pmod{p^{\alpha_p+1}} & \text{if } \alpha_p \geq 1 \\ n \equiv 2 \pmod{p} & \text{if } \alpha_p = 0 \end{cases}$$

Then, by Corollary 4.2,

$$-\deg \beta(\overline{a_{n+1}}) + \deg \beta(\overline{a_{n-1}}) = \sum_{\substack{r \in Q \\ r|q}} \text{mult}(r)$$

Step 2: Computation of  $\text{mult } 2$ .

Similarly, we may find  $n$  such that the elements of  $Q$  dividing  $n$  are exactly the ones dividing 60, such that 1 is the only element of  $Q$  dividing  $n-1$  or  $n+1$ , such that 1 and 2 are the only elements of  $Q$  dividing  $n-2$  (resp.  $n+2$ ).

Then, by Lemma 4.9,

$$-\deg \beta F^+(a_{n+2}) + \deg \beta F^+(a_{n-2}) = \text{mult } 2 + \sum_{\substack{r \in Q \\ r|60}} \text{mult}(r)$$

We may derive  $\text{mult } 2$  from the last since, in Step 1, we got an equation of the form

$$\sum_{\substack{r \in Q \\ r|60}} \text{mult}(r) = \text{cst}$$

where the right-hand side “cst” can be computed in terms of the  $\beta(\overline{a_i})$ 's.

Step 3: Computation of  $\text{mult } q$  for  $q \in Q$  such that  $2 \nmid q, 3 \nmid q$ .

Let  $q = \prod_{p \in \mathcal{P}} p^{\alpha_p}$  with  $\alpha_2 = \alpha_3 = 0$  and  $0 \leq \alpha_p \leq \gamma_p$  for  $p \in \mathcal{P} \setminus \{2, 3\}$ . Similarly, we may find  $n$  such that the elements of  $Q$  dividing  $n$  are exactly the ones dividing  $q$ , such that the elements of  $Q$  dividing  $n-1$  are exactly the ones dividing 12 if  $\alpha_5 \geq 1$  or the ones dividing 60 if  $\alpha_5 = 0$ , such that 1 is the only element of  $Q$  dividing  $n-2$  and such that 1 and 2 are the only elements of  $Q$  dividing  $n+1$  (resp.  $n-3$ ).

Then, by Lemma 4.9,

$$-\deg \beta F^+(a_{n+1}) + \deg \beta F^+(a_{n-3}) = \text{mult } 2 + \sum_{\substack{r \in Q \\ r|n-1}} \text{mult}(r) + \sum_{\substack{r \in Q \\ r|q}} \text{mult}(r)$$

Since we already computed  $\text{mult } 2$  in Step 2 and since we already got, in Step 1, an equation of the form

$$\sum_{\substack{r \in Q \\ r|n-1}} \text{mult}(r) = \text{cst}$$

we derive an equation of the form

$$\sum_{\substack{r \in Q \\ r|q}} \text{mult}(r) = \text{cst}$$

Then we may compute recursively  $\text{mult } q$  for  $q \in Q$  such that  $2 \nmid q, 3 \nmid q$  by varying the  $a_p$ .

Step 4: Computation of  $\text{mult } 3$  and  $\text{mult } 4$ .

Similarly, we may find  $n$  such that the only elements of  $Q$  dividing  $n$  are 1, 2, 4, such that the only element of  $Q$  dividing  $n+1$  is 1, such that the only elements of  $Q$  dividing  $n-1$  are 1, 3, such that the only elements of  $Q$  dividing  $n-2$  are 1, 2, such that the only elements of  $Q$  dividing  $n+2$  are exactly the ones dividing 210 and such that no element of  $Q$  divides  $n-3$  (resp.  $n+3$ ).

Then, by Corollary 4.2,

$$-\deg \beta F^+(a_{n+3}) + \deg \beta F^+(a_{n-3}) = 2\text{mult } 2 + \text{mult } 3 + \text{mult } 4 + \sum_{\substack{r \in Q \\ r|n+2}} \text{mult}(r)$$

Since we computed  $\text{mult } 2$  in Step 2 and since we get an equation of the form

$$\sum_{\substack{r \in Q \\ r|n+2}} \text{mult}(r) = \text{cst}$$

in Step 1, we get an equation of the form

$$(1) \quad \text{mult } 3 + \text{mult } 4 = \text{cst}$$

Now let  $n$  be such that the only elements of  $Q$  dividing  $n$  are the ones dividing 2520, the only element of  $Q$  dividing  $n-1$  (resp.  $n+1$ ) is 1, the only elements of  $Q$  dividing  $n-2$  (resp.  $n+2$ ) are 1, 2, the only elements of  $Q$  dividing  $n-3$  (resp.  $n+3$ ) are 1, 3 and such that the only elements of  $Q$  dividing  $n-4$  (resp.  $n+4$ ) are 1, 2, 4.

Then, by Lemma 4.10,

$$-\deg \beta F^+(a_{n+4}) + \deg \beta F^+(a_{n-4}) = \sum_{\substack{r \in Q \\ r|n}} \text{mult}(r) + 3\text{mult } 2 + 2\text{mult } 3 + \text{mult } 4$$

Since we computed  $\text{mult } 2$  in Step 2 and since we get an equation of the form

$$\sum_{\substack{r \in Q \\ r|n}} \text{mult}(r) = \text{cst}$$

in Step 1, we get an equation of the form

$$(2) \quad 2\text{mult } 3 + \text{mult } 4 = \text{cst}$$

Equations (1) and (2) allow us to compute  $\text{mult } 3$  and  $\text{mult } 4$ .

Step 5: Computation of  $\text{mult } q$  for  $q \in Q$  such that  $2|q, 3 \nmid q, 4 \nmid q$ .

Let  $q = \prod_{p \in \mathcal{P}} p^{\alpha_p}$  with  $\alpha_2 = 1, \alpha_3 = 0$  and  $0 \leq \alpha_p \leq \gamma_p$  for  $p \in \mathcal{P} \setminus \{2, 3\}$ . Similarly, we may find  $n$  such that the elements of  $Q$  dividing  $n$  are exactly the ones dividing  $q$ , such that the elements of  $Q$  dividing  $n-1$  are  $1, 3$ , such that the elements of  $Q$  dividing  $n-2$  are  $1, 2, 4$ , such that the elements of  $Q$  dividing  $n+2$  are the ones dividing  $84$ , such that  $1$  is the only element dividing  $n-3$  (resp.  $n+3$ ) and such that the only elements of  $Q$  dividing  $n+1$  are  $1$  if  $\alpha_5 \geq 1$  or  $1, 5$  if  $\alpha_5 = 0$ .

Then, by Corollary 4.2,

$$-\deg \beta(\overline{a_{n+3}}) + \deg \beta(\overline{a_{n-3}}) = \sum_{\substack{r \in Q \\ r|q}} \text{mult}(r) + \sum_{\substack{r \in Q \\ r|n+2}} \text{mult}(r) + \text{mult } 2 + \text{mult } 3 + \text{mult } 4 (+ \text{mult } 5)$$

Since we already computed  $\text{mult } 2$  in Step 2,  $\text{mult } 3, \text{mult } 4$  in Step 4,  $\text{mult } 5$  in Step 3 and since we got in Step 1 an equation of the form

$$\sum_{\substack{r \in Q \\ r|n+2}} \text{mult}(r) = \text{cst}$$

we finally obtain an equation of the form

$$\sum_{\substack{r \in Q \\ r|q}} \text{mult}(r) = \text{cst}$$

Thus we may compute  $\text{mult } q$  for  $q \in Q$  such that  $2|q, 3 \nmid q, 4 \nmid q$  by varying the  $\alpha_p$ .

Step 6: Computation of  $\text{mult } q$  for  $q \in Q$  such that  $4|q, 3 \nmid q$ .

Let  $q = \prod_{p \in \mathcal{P}} p^{\alpha_p}$  with  $\alpha_2 \geq 2, \alpha_3 = 0$  and  $0 \leq \alpha_p \leq \gamma_p$  for  $p \in \mathcal{P} \setminus \{2, 3\}$ . Similarly, we may find  $n$  such that the elements of  $Q$  dividing  $n$  are exactly the ones dividing  $q$ , such that the elements of  $Q$  dividing  $n-1$  are  $1, 3$ , such that the elements dividing  $n-2$  are  $1, 2$  if  $\alpha_5 \geq 1$  or  $1, 2, 5, 10$  if  $\alpha_5 = 0$  and such that  $1$  is the only element in  $Q$  dividing  $n-3$  (resp.  $n+1$ ).

Then, by Corollary 4.2,

$$-\deg \beta(\overline{a_{n+1}}) + \deg \beta(\overline{a_{n-3}}) = \sum_{\substack{r \in Q \\ r|q}} \text{mult}(r) + \sum_{\substack{r \in Q \\ r|n-2}} \text{mult}(r) + \text{mult } 3$$

Since we already computed  $\text{mult } 3$  in Step 4 and since we got an equation of the form  $\sum_{\substack{r \in Q \\ r|n-2}} \text{mult}(r)$  in Step 5, we obtain an equation of the form

$$\sum_{\substack{r \in Q \\ r|q}} \text{mult}(r) = \text{cst}$$

Thus we may compute  $\text{mult } q$  for  $q \in Q$  such that  $4|q, 3 \nmid q$  by varying the  $\alpha_p$ .

Step 7: Computation of  $\text{mult } q$  for  $q \in Q$  such that  $3|q, 2 \nmid q$ .

Let  $q = \prod_{p \in \mathcal{P}} p^{\alpha_p}$  with  $\alpha_3 \geq 1, \alpha_2 = 0$  and  $0 \leq \alpha_p \leq \gamma_p$  for  $p \in \mathcal{P} \setminus \{2, 3\}$ . Similarly, we may find  $n$  such that the elements of  $Q$  dividing  $n$  are exactly the ones dividing  $q$ , such that the elements of  $Q$  dividing  $n-1$  are  $1, 2, 4$ , such that the elements dividing  $n+1$  are  $1, 2$  if  $\alpha_5 \geq 1$  or  $1, 2, 5, 10$  if  $\alpha_5 = 0$  and such that  $1$  is the only element in  $Q$  dividing  $n-2$  (resp.  $n+2$ ).

Then, by Corollary 4.2,

$$-\deg \beta(\overline{a_{n+2}}) + \deg \beta(\overline{a_{n-2}}) = \sum_{\substack{r \in Q \\ r|q}} \text{mult}(r) + \sum_{\substack{r \in Q \\ r|n-1}} \text{mult}(r) + \sum_{\substack{r \in Q \\ r|n+1}} \text{mult}(r)$$

Since we got an equation of the form  $\sum_{\substack{r \in Q \\ r|n-1}} \text{mult}(r)$  in **Step 6** and an equation of the form  $\sum_{\substack{r \in Q \\ r|n+1}} \text{mult}(r)$  in **Step 5**, we obtain an equation of the form

$$\sum_{\substack{r \in Q \\ r|q}} \text{mult}(r) = \text{cst}$$

Thus we may compute  $\text{mult } q$  for  $q \in Q$  such that  $3|q, 2 \nmid q$  by varying the  $\alpha_p$ .

**Step 8:** Computation of  $\text{mult}(q)$  for  $q \in Q$  such that  $6|q$ .

We may now compute  $\text{mult}(q)$  for  $q \in Q$  such that  $6|q$  by varying the  $\alpha_p$  in **Step 1**.

■

The arguments of the previous proof also allow one to prove [2, Corollary 8.4] without using the convolution formula.

**Theorem 4.12** ([2, Corollary 8.4]). *The arc-analytic type of a singular Brieskorn polynomial determines its exponents.*

## References

- [1] O. M. ABDERRAHMANE, *Weighted homogeneous polynomials and blow-analytic equivalence*, in Singularity theory and its applications, vol. 43 of Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, 2006, pp. 333–345.
- [2] J.-B. CAMPESATO, *On a motivic invariant of the arc-analytic equivalence*, 2015, arXiv:1512.07145. To appear in Annales de l’Institut Fourier.
- [3] ———, *From the blow-analytic equivalence to the arc-analytic equivalence: a survey*, 2016. To appear in the Proceedings of JARCS6 (Sixth Japanese-Australian Workshop on Real & Complex Singularities) in Saitama Mathematical Journal.
- [4] ———, *An inverse mapping theorem for blow-Nash maps on singular spaces*, Nagoya Math. J., 223 (2016), pp. 162–194.
- [5] J. DENEFF AND K. HOORNAERT, *Newton polyhedra and Igusa’s local zeta function*, J. Number Theory, 89 (2001), pp. 31–64.
- [6] J. DENEFF AND F. LOESER, *Caractéristiques d’Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc., 5 (1992), pp. 705–720.
- [7] ———, *Motivic Igusa zeta functions*, J. Algebraic Geom., 7 (1998), pp. 505–537.
- [8] G. FICHOUE, *Motivic invariants of arc-symmetric sets and blow-Nash equivalence*, Compos. Math., 141 (2005), pp. 655–688.
- [9] ———, *Zeta functions and blow-Nash equivalence*, Ann. Polon. Math., 87 (2005), pp. 111–126.
- [10] ———, *The corank and the index are blow-Nash invariants*, Kodai Math. J., 29 (2006), pp. 31–40.
- [11] ———, *Towards a classification of blow-Nash types*, in Singularities and o-minimal category, RIMS Kôkyûroku 1540, Kyoto University, 04 2007, pp. 145–151.
- [12] G. FICHOUE AND T. FUKUI, *Motivic invariants of real polynomial functions and their Newton polyhedrons*, Math. Proc. Cambridge Philos. Soc., 160 (2016), pp. 141–166.
- [13] T. FUKUI, *Seeking invariants for blow-analytic equivalence*, Compositio Math., 105 (1997), pp. 95–108.
- [14] G. GUIBERT, *Espaces d’arcs et invariants d’Alexander*, Comment. Math. Helv., 77 (2002), pp. 783–820.
- [15] G. GUIBERT, F. LOESER, AND M. MERLE, *Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink*, Duke Math. J., 132 (2006), pp. 409–457.
- [16] A. G. HOVANSKIĬ, *Newton polyhedra, and the genus of complete intersections*, Funktsional. Anal. i Prilozhen., 12 (1978), pp. 51–61.
- [17] S. KOIKE AND A. PARUSIŃSKI, *Motivic-type invariants of blow-analytic equivalence*, Ann. Inst. Fourier (Grenoble), 53 (2003), pp. 2061–2104.
- [18] A. G. KOUCHNIRENKO, *Polyèdres de Newton et nombres de Milnor*, Invent. Math., 32 (1976), pp. 1–31.
- [19] T.-C. KUO, *On classification of real singularities*, Invent. Math., 82 (1985), pp. 257–262.
- [20] K. KURDYKA, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann., 282 (1988), pp. 445–462.
- [21] C. MCCRORY AND A. PARUSIŃSKI, *Virtual Betti numbers of real algebraic varieties*, C. R. Math. Acad. Sci. Paris, 336 (2003), pp. 763–768.

- [22] ———, *The weight filtration for real algebraic varieties*, in *Topology of stratified spaces*, vol. 58 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 2011, pp. 121–160.
- [23] J. MILNOR, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.
- [24] A. PARUSIŃSKI, *Topology of injective endomorphisms of real algebraic sets*, Math. Ann., 328 (2004), pp. 353–372.
- [25] A. PARUSIŃSKI AND L. PAUNESCU, *Arcwise Analytic Stratification, Whitney Fibering Conjecture and Zariski Equisingularity*, ArXiv e-prints, (2015), 1503.00130.
- [26] M. RAIBAUT, *Singularités à l’infini et intégration motivique*, Bull. Soc. Math. France, 140 (2012), pp. 51–100.
- [27] O. SAEKI, *Topological invariance of weights for weighted homogeneous isolated singularities in  $\mathbb{C}^3$* , Proc. Amer. Math. Soc., 103 (1988), pp. 905–909.
- [28] K. SAITO, *Quasihomogene isolierte Singularitäten von Hyperflächen*, Invent. Math., 14 (1971), pp. 123–142.
- [29] A. N. VARCHENKO, *Zeta-function of monodromy and Newton’s diagram*, Invent. Math., 37 (1976), pp. 253–262.
- [30] E. YOSHINAGA AND M. SUZUKI, *Topological types of quasihomogeneous singularities in  $\mathbb{C}^2$* , Topology, 18 (1979), pp. 113–116.